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General teleportation as a quantum channel

Yong-Jian Gu, Chun-Mei Yao, Zheng-Wei Zhou and Guang-Can Guo

Key Laboratory of Quantum Information, University of Science and Technology of China, CAS, Hefei 230026, People's Republic of China

E-mail: yjgu62@mail.ustc.edu.cn and gcguo@ustc.edu.cn

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Abstract

Teleportation is viewed as a quantum channel. We present an explicit expression for the general teleportation channel in the Kraus decomposition form. We then analyse optimal teleportation procedures for a noisy entangled resource, Bell measurement by the sender and arbitrary operations by the receiver. Our general result allows us to derive the corresponding quantum channel and fidelity, thereby enabling us to formulate the fidelity-based optimization problem and to conclude that this is a problem of semidefinite programming. We offer an alternative viewpoint on optimal teleportation, namely one can perform corrective operations at the receiver's side instead of first manipulating entanglement, and we give an optimal teleportation strategy.

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1. Introduction

Quantum teleportation [1], one of the most fascinating discoveries in quantum information science, plays a key role in quantum communication and quantum computation [2]. Teleportation can naturally be related to quantum channels since it is a process of transmission of an unknown quantum state, although the transmission is implemented not by directly sending particles through a channel but via local operations, classical communication and shared entanglement (LOCCSE). Mathematically, a quantum channel is described by a completely positive trace-preserving (CPTP) linear map that maps an input state onto an output one: $\rho' = \mathcal{E}(\rho)$. The map $\mathcal{E}$, also called a quantum operation, can be represented in the form [2, 3] $\mathcal{E}(\rho) = \sum_j A_j \rho A_j^\dagger$ with $\sum_j A_j^\dagger A_j = I$, known as the operator-sum representation or Kraus decomposition.

The nature of the teleportation channel is dependent on both the entangled state resource and the particular LOCC protocol [4–6]. Recently it was shown that the standard teleportation protocol (Bell measurement and corresponding Pauli rotations) with a mixed entangled resource is equivalent to a generalized depolarizing channel [7]. This result was then generalized to continuous variable teleportation [8] and the situation where the receiver is allowed to perform arbitrary unitary operations [9]. A natural question to ask is: how to formulate a general teleportation process in terms of a quantum channel? By general teleportation we mean the generalization of the standard teleportation to an arbitrary mixed entangled state as the resource, a generalized measurement on the sender's side and corresponding CPTP operations on the receiver's side. This generalization is important in that for practical purposes, one has to consider various kinds of non-idealities and try to design corresponding strategies to conquer them [11–17]. Nielsen and Caves [10] formulated the actions of the resource and the sender's measurement as a quantum operation, regarding the receiver's operations as reversing the quantum operation. In section 2, we present an explicit expression for the general teleportation channel by adapting the Nielsen–Caves formalism to formulate the whole teleportation process. This completely generalizes the result of [7].

In section 3, we consider the following problem: given an arbitrary resource, what are the receiver's operations that maximize the fidelity provided that the sender is restricted to performing a Bell measurement? Using the general result in section 2 for the special case, we obtain the corresponding quantum channel and fidelity. This allows us to express the fidelity-based optimization problem and to conclude that this is a problem of semidefinite programming. Our results offer an alternative viewpoint, namely in order to achieve optimal teleportation one can perform manipulation at the receiver's side after the measurement, instead of first manipulating the entangled state [18]; and we give an optimal teleportation strategy.

2. Explicit expression for the general teleportation channel

Suppose that the sender Alice and the receiver Bob share two particles in an arbitrary mixed entangled state τ_{AB}, where A and B stand for the particles on Alice's and Bob's sides, respectively, and Alice is given a particle Q in an unknown quantum state $\bar{\rho}_Q$ to be teleported to Bob, each particle with the same d-dimensional state space. Thus the composite system is initially in the state $\bar{\rho}_Q \otimes \tau_{AB}$. To start teleportation, Alice performs a measurement on particles Q and A, which is described by a positive operator-valued measure (POVM)

$$\Pi^i_{QA} = A^{i\dagger}_{QA} A^i_{QA} \qquad \sum_i \Pi^i_{QA} = I_{QA} \tag{1}$$

the index i labelling the outcomes of the measurement. The state of Bob's particle B conditioned on the outcome i is given by

$$\rho^i_B = \frac{1}{p_i} \mathrm{tr}_{QA} \left[\left(A^i_{QA} \otimes I_B \right) (\bar{\rho}_Q \otimes \tau_{AB}) \left(A^{i\dagger}_{QA} \otimes I_B \right) \right] \tag{2}$$

where

$$p_i = \mathrm{tr}_{QAB} \left[\left(A^i_{QA} \otimes I_B \right) (\bar{\rho}_Q \otimes \tau_{AB}) \left(A^{i\dagger}_{QA} \otimes I_B \right) \right]$$

is the probability of obtaining the outcome i. Then Bob applies an i-dependent transformation on his particle. The most general transformation is a CPTP map of the form

$$\mathcal{E}^i \left(\rho^i_B \right) = \sum_j D^{ij}_B \rho^i_B D^{ij\dagger}_B \qquad \sum_j D^{ij\dagger}_B D^{ij}_B = I_B. \tag{3}$$

Therefore, over all outcomes i, the final total teleported state reads

$$\rho'_B = \sum_i p_i \mathcal{E}^i \left(\rho^i_B \right) = \sum_{i,j} D^{ij}_B \, \mathrm{tr}_{QA} \left[\left(A^i_{QA} \otimes I_B \right) (\bar{\rho}_Q \otimes \tau_{AB}) \left(A^{i\dagger}_{QA} \otimes I_B \right) \right] D^{ij\dagger}_B. \tag{4}$$

Now a trick will be applied. Following the line in [10], we introduce the swap operator U_{QB} which swaps the states of particles Q and B, while leaving particle A alone. Clearly, $U_{QB} = U_{QB}^{-1} = U_{QB}^{\dagger}$. By using the swap operator we have

$$\bar{\rho}_Q \otimes \tau_{AB} = U_{QB}(\bar{\tau}_{QA} \otimes \rho_B)U_{QB} \tag{5}$$

where ρ_B ($\bar{\tau}_{QA}$) is the counterpart of $\bar{\rho}_Q$ (τ_{AB}), i.e., ρ_B is exactly the same state of particle B as the one of particle Q. We expand $\bar{\tau}_{QA}$ in the complete orthonormal set of its eigenstates $\left|\bar{s}_{QA}^k\right\rangle$

$$\bar{\tau}_{QA} = \sum_k r_k \left|\bar{s}_{QA}^k\right\rangle\left\langle\bar{s}_{QA}^k\right|. \tag{6}$$

Substituting equations (5) and (6) into equation (4) and performing the partial trace in any complete orthonormal basis $\left|P_{QA}^l\right\rangle$ for the joint system Q and A yields

$$\rho_B' = \mathcal{E}(\rho_B) = \sum_{i,j,k,l} M_B^{ijkl} \rho_B M_B^{ijkl\dagger} \tag{7}$$

where the operator M_B^{ijkl} acting on particle B is given by

$$M_B^{ijkl} = \sqrt{r_k} D_B^{ij} \left\langle P_{QA}^l\right| \left(A_{QA}^i \otimes I_B\right) U_{QB} \left|\bar{s}_{QA}^k\right\rangle. \tag{8}$$

Using the completeness relations in (1) and (3), it is easy to check that

$$\sum_{i,j,k,l} M_B^{ijkl\dagger} M_B^{ijkl} = I_B. \tag{9}$$

Thus we have obtained an explicit expression for a general teleportation channel, the actions of the entangled resource and the LOCC protocol being reflected by the Kraus operators M_B^{ijkl}. Because of its generality, this result enables one to obtain the quantum channel of particular teleportation processes.

3. Optimal teleportation via Bell measurement

For simplicity, we next restrict our attention to the two-dimensional state space, although our discussions can naturally be generalized to the high-dimensional case by using the generalized Pauli operators. Let us now study the teleportation via the Bell measurement on the sender's side and corresponding CPTP operations on the receiver's side. In this case we have $A_{QA}^i = \left|\Phi_{QA}^i\right\rangle\left\langle\Phi_{QA}^i\right|$, where $\left|\Phi_{QA}^i\right\rangle$ $(i = 0, 1, 2, 3)$ are the Bell states associated with the Pauli operators $\sigma^0 = I, \sigma^1 = \sigma_x, \sigma^2 = \sigma_y, \sigma^3 = \sigma_z$ by $\left|\Phi_{QA}^i\right\rangle = \left(\sigma_Q^i \otimes \sigma_A^0\right)|\Phi\rangle$, with $|\Phi\rangle$ one of the maximally entangled Bell states. We make the choice $\left|P_{QA}^l\right\rangle = \left|\Phi_{QA}^l\right\rangle$. Then equation (7) reduces to

$$\rho_B' = \sum_{i,j,k} r_k D_B^{ij} \left\langle\Phi_{QA}^i\right| U_{QB} \left|\bar{s}_{QA}^k\right\rangle \rho_B \left\langle\bar{s}_{QA}^k\right| U_{QB} \left|\Phi_{QA}^i\right\rangle D_B^{ij\dagger} \tag{10}$$

where we have used $\left\langle\Phi_{QA}^l\middle|\Phi_{QA}^i\right\rangle = \delta_{li}$. It is easy to give that

$$U_{QB} = \frac{1}{2}\sum_m \sigma_Q^m \otimes \sigma_A^0 \otimes \sigma_B^m. \tag{11}$$

We expand $\left|\bar{s}_{QA}^k\right\rangle$ in the Bell basis:

$$\left|\bar{s}_{QA}^k\right\rangle = \sum_n s_{kn} \left|\Phi_{QA}^n\right\rangle. \tag{12}$$

Let the operator $\left\langle \Phi^0_{QA} \right| U_{QB} \left| \bar{s}^k_{QA} \right\rangle$ acting on particle B be denoted by E^k_B. Using the above expressions and the properties of Pauli operators, we obtain

$$E^k_B = \frac{1}{2}\sum_n s_{kn}\sigma^n_B \tag{13}$$

$$\left\langle \Phi^i_{QA} \right| U_{QB} \left| \bar{s}^k_{QA} \right\rangle = E^k_B \sigma^i_B. \tag{14}$$

We finally get

$$\begin{aligned}
\rho' &= \sum_{i,j,k} r_k D^{ij} E^k \sigma^i \rho \sigma^i E^{k\dagger} D^{ij\dagger} \\
&= \frac{1}{4}\sum_{i,j,k,n,n'} r_k s_{kn} s^*_{kn'} D^{ij} \sigma^n \sigma^i \rho \sigma^i \sigma^{n'} D^{ij\dagger} \\
&= \frac{1}{4}\sum_{n,n'} \langle \Phi^n | \tau | \Phi^{n'} \rangle \sum_{i,j} D^{ij} \sigma^n \sigma^i \rho \sigma^i \sigma^{n'} D^{ij\dagger}
\end{aligned} \tag{15}$$

where we have used $\sum_k r_k s_{kn} s^*_{kn'} = \langle \Phi^n | \tau | \Phi^{n'} \rangle$, and we have dropped the subscripts Q, A, B. Equation (15) is the explicit expression for the quantum channel of the teleportation via Bell measurement. If Bob's operations are unitary, namely $D^{ij} = U^i$, then we have

$$\rho' = \frac{1}{4}\sum_{n,n'} \langle \Phi^n | \tau | \Phi^{n'} \rangle \sum_i U^i \sigma^n \sigma^i \rho \sigma^i \sigma^{n'} U^{i\dagger}. \tag{16}$$

This is just the result in [9]. Specifically, if $U^i = \sigma^i$, i.e., the standard teleportation is performed, equation (16) reduces to

$$\rho' = \frac{1}{4}\sum_{n,n'} \langle \Phi^n | \tau | \Phi^{n'} \rangle \sum_i \sigma^i \sigma^n \sigma^i \rho \sigma^i \sigma^{n'} \sigma^i \tag{17}$$

$$= \sum_n \langle \Phi^n | \tau | \Phi^n \rangle \sigma^n \rho \sigma^n. \tag{18}$$

The equality between equations (17) and (18) can be shown by noting that all terms except those of the form given in equation (18) cancel. From equation (18), it follows that the general teleportation channel reduces to the generalized depolarizing channel when teleportation is performed with an arbitrary mixed entangled resource under the standard protocol. Here we have presented another proof of the important conclusion as a special case of our general discussion.

The explicit input–output relation (15) makes it easy to calculate the fidelity of teleportation. Suppose that the to-be-teleported pure state is $\rho = |\psi\rangle\langle\psi|$. Then the teleportation fidelity is

$$\begin{aligned}
F &= \overline{\langle \psi | \rho' | \psi \rangle} \\
&= \frac{1}{4}\sum_{n,n'} \langle \Phi^n | \tau | \Phi^{n'} \rangle \sum_{i,j} \overline{\langle \psi | D^{ij} \sigma^n \sigma^i | \psi \rangle \langle \psi | \sigma^i \sigma^{n'} D^{ij\dagger} | \psi \rangle} \\
&= \frac{1}{4}\sum_{n,n'} \langle \Phi^n | \tau | \Phi^{n'} \rangle \sum_{i,j} \overline{\langle \psi | \langle \psi | D^{ij} \sigma^n \sigma^i \otimes \sigma^i \sigma^{n'} D^{ij\dagger} | \psi \rangle | \psi \rangle}
\end{aligned} \tag{19}$$

where the average is taken over the isotropic *a priori* distribution of states to be teleported. In order to calculate the fidelity F, we need an irreducible n-dimensional representation of the unitary group $\mathbf{U(n)}$, denoted by $\mathbf{G}$. Let $U(g)$ be the unitary matrix representation of the element g of $\mathbf{G}$. Recalling Schur's lemma, we have the identity

$$\int_{\mathbf{G}} \mathrm{d}g (U^\dagger(g) \otimes U^\dagger(g))\sigma(U(g) \otimes U(g)) = \alpha_1 I \otimes I + \alpha_2 P$$

with

$$\alpha_1 = \frac{n^2 \operatorname{tr}(\sigma) - n \operatorname{tr}(\sigma P)}{n^2(n^2-1)} \qquad \alpha_2 = \frac{n^2 \operatorname{tr}(\sigma P) - n \operatorname{tr}(\sigma)}{n^2(n^2-1)}$$

for any operator σ acting on the tensor space, where P is an operator such that $P\,|ij\rangle = |ji\rangle$. The invariant (Haar) measure $\mathrm{d}g$ on $\mathbf{G}$ is normalized by $\int_{\mathbf{G}} \mathrm{d}g = 1$. Thus $\overline{(\cdot)}$ in equation (19) can be calculated as follows,

$$\overline{(\cdot)} = \langle 00| \int_{\mathbf{G}} \mathrm{d}g (U^\dagger(g) \otimes U^\dagger(g))(D^{ij}\sigma^n\sigma^i \otimes \sigma^i\sigma^{n'}D^{ij\dagger})\,(U(g) \otimes U(g))\,|00\rangle$$
$$= \frac{1}{6}[\operatorname{tr}(D^{ij}\sigma^n\sigma^i)\operatorname{tr}(\sigma^i\sigma^{n'}D^{ij\dagger}) + \operatorname{tr}(\sigma^n\sigma^{n'}D^{ij\dagger}D^{ij})]$$

where the identity $\operatorname{tr}_{12}((A \otimes B)P) = \operatorname{tr}(AB)$ has been used. Using the completeness relation in (3) and the fact that any operator χ can be expanded as

$$\chi = \frac{1}{2}\sum_i \operatorname{tr}(\sigma^i\chi)\sigma^i$$

we finally arrive at

$$F = \frac{1}{6}\langle\Phi| \left[\sum_{i,j}(\sigma^i D^{ij} \otimes I)\tau(D^{ij\dagger}\sigma^i \otimes I)\right]|\Phi\rangle + \frac{1}{3} = \frac{2f+1}{3} \tag{20}$$

$$f = \frac{1}{4}\langle\Phi| \left[\sum_{i,j}(\sigma^i D^{ij} \otimes I)\tau(D^{ij\dagger}\sigma^i \otimes I)\right]|\Phi\rangle\,. \tag{21}$$

Note that when Pauli rotations are performed by Bob, i.e., $D^{ij} = \sigma^i$, then f reduces to the singlet fraction $\langle\Phi|\tau|\Phi\rangle$ of the resource τ. Also, when optimal unitary operations are performed by Bob, i.e., $D^{ij} = U^i$, with U^i unitary and $\sigma^i U^i = W$, where W is a unitary operator such that $\langle\Phi|(W \otimes I)\tau(W^\dagger \otimes I)|\Phi\rangle$ is the fully entangled fraction [19] of the state τ, then f reduces to the fully entangled fraction of the state τ.

There is no reason to believe that unitary operations represent the most efficient teleportation strategy. In a surprising paper [18], Badziag *et al* proved that a non-unitary CPTP map may provide better performance of teleportation than a unitary operation by showing that for a certain entangled resource, a non-unitary amplitude-damping operation enables better than classical teleportation even when a unitary operation does not. So it is important to find the optimal CPTP maps which maximize teleportation fidelity. We now come to this problem.

Since CPTP maps corresponding to different measurement results i are independent, each term in equation (21) can be maximized independently. Let us first consider the problem of maximizing

$$f' = \langle\Phi|(\mathcal{E} \otimes I)(\tau)|\Phi\rangle = \langle\Phi| \left[\sum_j (C^j \otimes I)\tau(C^{j\dagger} \otimes I)\right]|\Phi\rangle \tag{22}$$

where C^j are the Kraus operators of the map $\mathcal{E}$. Using the isomorphism [20] between the CPTP map and the positive operator τ' of the bipartite system with one subsystem maximally mixed, we can obtain [18] $f' = \mathrm{tr}(\tau\tau')$. Thus the (state-dependent) maximization problem can be formulated as

$$\text{maximize}\,\mathrm{tr}(\tau\tau') \qquad \tau' \geqslant 0 \qquad \mathrm{tr}_2\,\tau' = I_1. \tag{23}$$

Here we would like to point out that this is just a problem of semidefinite programming [21] and can easily be solved numerically with guaranteed convergence. The required knowledge of the density matrix τ can be found experimentally with quantum tomography [22, 23]. Having obtained the operator τ', one can obtain C^j [24]. Then by adopting the strategy $D^{ij} = \sigma^i C^j$ for all i, f in equation (21) is maximized, and the maximum of f is given by

$$f_{\max} = \langle\Phi| \left[\sum_j (C^j \otimes I)\tau(C^{j\dagger} \otimes I)\right] |\Phi\rangle\,. \tag{24}$$

This is in agreement with the result on optimal teleportation [4, 18]. Namely our strategy is optimal. The meaning of this strategy is that after receiving Alice's measurement result i, Bob first performs on his particle the CPTP operation denoted by C^j, then performs a corresponding Pauli rotation. This strategy of manipulating a particle not entangled with others is fundamentally different from the scheme of first manipulating the entanglement [18].

To be clear, we formulate the main results in this section in the following theorem:

Theorem. *The optimal teleportation via a single copy of an arbitrary mixed entangled resource τ, the sender's Bell measurement with the result i and the receiver's corresponding CPTP operation described by the Kraus operators $\sigma^i C^j$, acts as a quantum channel*

$$\rho' = \frac{1}{4}\sum_{n,n'}\langle\Phi^n|\tau|\Phi^{n'}\rangle \sum_{i,j}\sigma^i C^j \sigma^n \sigma^i \rho \sigma^i \sigma^{n'} C^{j\dagger}\sigma^i \tag{25}$$

where $|\Phi^n\rangle$ are the Bell states. The operators C^j can be found via semidefinite programming

$$\text{maximize}\,\mathrm{tr}(\tau\tau') \qquad \tau' \geqslant 0 \qquad \mathrm{tr}_2\,\tau' = I_1$$

and the isomorphism between the CPTP map and the positive operator τ'.

4. Conclusion

We have presented an explicit expression for the general teleportation channel. Using the general result as a starting point, we have studied the teleportation via an arbitrary mixed entangled resource, a Bell measurement by the sender and corresponding completely positive trace-preserving operations by the receiver. As a result, we have derived the corresponding quantum channel and the fidelity of teleportation. By maximizing the fidelity via semidefinite programming, we have constructed an optimal teleportation strategy. We have offered an alternative viewpoint on optimal teleportation, namely one can perform corrective operations on the receiver's side instead of first manipulating entanglement. It should be pointed out here that these conclusions do not necessarily apply when the sender performs more general measurements.

Regarding teleportation as a quantum channel provides one with an approach to study teleportation. The power of this approach lies in the fact that it in principle allows one to derive all results about a teleportation process, such as fidelity, since the expression for the teleportation channel in the Kraus decomposition form provides an explicit input–output relation. In the above we have illustrated the power of this approach. In principle, different combinations of entangled resources and LOCC protocols result in different quantum channels. In addition to the case discussed above, other teleportation processes may also be discussed from our general expression.

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